DIFFRACTION OF A PLANE WAVE BY A FINE PERIODIC GRID PLACED AT THE INTERFACE OF TWO MEDIA

PMM Vol. 34, №3, 1970, pp. 543-547
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(Received January 5, 1970)

Using the method proposed in [1] we determine the long-wave asymptotic behavior of the coefficients of reflection and transmission of a plane wave incident on a grid placed at the interface of two media. We prove that this asymptotic behavior is fully expressed by the apparent mass of the grid.

Let us have a $2c$-periodic grid placed in the plane xy along the axis y. The elements of the grid are convex domains with two mutually perpendicular axes of symmetry, one of them being axis y. We shall denote by D the domain outside the grid, by D^+ that part of D which lies in half-plane $x > 0$, and by D^- that part of D which corresponds to $x < 0$ (see Fig. 1).

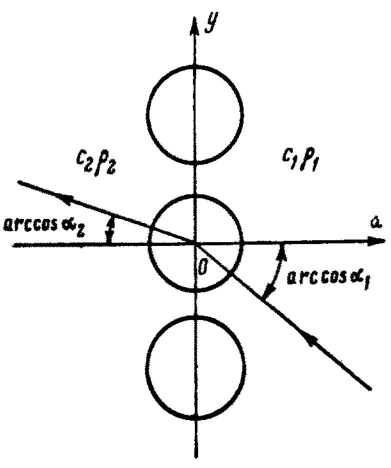

Fig. 1

Let us also assume that a plane wave

$$P_0 = e^{-ik_1(\alpha_1 x - \beta_1 y)}, \quad \alpha_1^2 + \beta_1^2 = 1, \quad k_1 = \omega / c_1$$

incident on the grid, arrives from the right-hand side $(x > 0)$. In the above equation k_1 is the wave number, ω is the frequency of incident field, c_1 is the wave velocity, and α_1 and β_1 are the cosines defining the propagation direction of the wave.

We shall consider a fine grid, i. e. period $2c$ of the grid is taken to be many times smaller than k_1/ω ($|k_1| < \pi/2c$).

The problem of diffraction of P_0 on this grid consists in finding a function $P(x, y; k)$, regular with respect to k for $|k| < \pi / 2c$ and, in the domain D, satisfying:

the Helmholtz equation

$$(\Delta + k^2) P = 0 \tag{1}$$

the homogeneous Neumann condition at the grid boundary γ

$$\left.\frac{\partial P}{\partial n}\right|_\gamma = 0 \tag{2}$$

the condition of quasi-periodicity in domain D

$$P(x,y) = P(x, y - 2c) e^{2ikc\beta} \tag{3}$$

the decay condition formulated by Maliuzhinets [1]

$$\sup_D |(P - P_0) e^{-ik\beta y}| < \infty \quad \text{for} \quad \operatorname{Im} k > |\operatorname{Re} k| \tag{4}$$

the conditions on L (the totality of the y-axis sections located in D)

$$P_1 = P_2 \quad \text{(continuity of pressure on } L) \\ \frac{1}{\rho_1}\frac{\partial P_1}{\partial x} = \frac{1}{\rho_2}\frac{\partial P_2}{\partial x} \quad \text{(continuity of velocity on } L) \tag{5}$$

Constants ρ_1 and ρ_2 are characteristics of densities of the media in domains D^+ and D^-. Symbol n denotes a normal to γ and is inside D, and

$$P = \begin{cases} P_1(x, y; k) & (x, y) \in D^+ \\ P_2(x, y; k) & (x, y) \in D^- \end{cases} \qquad k = \begin{cases} k_1 = \omega / c_1 & (x, y) \in D^+ \\ k_2 = \omega / c_2 & (x, y) \in D^- \end{cases}$$

$$\beta = \begin{cases} \beta_1 & (x, y) \in D^+ \\ \beta_2 & (x, y) \in D^- \end{cases} \qquad \alpha = \begin{cases} \alpha_1 & (x, y) \in D^+ \\ \alpha_2 & (x, y) \in D^- \end{cases}$$

where c_2 is the wave propagation velocity in D^-.

Let us note that conditions (3) and (4) ensure the uniqueness of solution of the diffraction problem formulated in (1)–(5).

Let us determine the asymptotic behavior of P when $|x| \to \infty$ and $|k| < (\pi / 2c)$. With this purpose in mind, let us first point out that by virtue of (3), the function $Pe^{-ik\beta y}$ must be periodic with a period $2c$, so that the following expression is valid:

$$P = e^{ik\beta y} \sum_{\infty < n < \infty}' C_n(x) \exp \frac{i\pi n y}{c} \tag{6}$$

Substituting this series into (1), we obtain an ordinary differential equation for each of the functions $C_n(x)$ $\quad d^2C_n / dx^2 - [(\pi n / c)\lambda_n]^2 C_n = 0$

where

$$(\pi n / c)\lambda_n = \sqrt{[(\pi n / c) + k\beta]^2 - k^2} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

the function $\sqrt{\lambda_n^2}$ ($|n| > 0$) is determined in the plane of complex variable with a cut along the ray $\operatorname{Re}\lambda_n^2 < 0$, $\operatorname{Im}\lambda_n^2 = 0$, and assumes the value $\sqrt{\lambda_n^2} = 1$ when $k = 0$.

Straight calculation yields the following inequality:

$$\operatorname{Re} \lambda_n > 0 \quad \text{for} \quad |n| > 0, \ |k| < (\pi / 2c)$$

which allowing for (4), makes it possible to transform (6) as follows:

$$Pe^{-ik\beta y} = ae^{-ik\alpha x} + be^{ik\alpha x} +$$
$$+ \sum_{n=1}^{\infty}{}' a_n \exp\left\{\frac{\pi n}{c}\left[-|x|\left[\left(1 - \frac{kc}{\pi n}\beta\right)^2 - \left(\frac{kc}{\pi n}\right)^2\right]^{1/2} - iy\right]\right\} +$$
$$+ \sum_{n=1}^{\infty}{}' b_n \exp\left\{\frac{\pi n}{c}\left[-|x|\left[\left(1 + \beta\frac{kc}{\pi n}\right)^2 - \left(\frac{kc}{\pi n}\right)^2\right]^{1/2} + iy\right]\right\}$$

Taking now into account that for $|k| < (\pi / 2c)$ and $|n| > 0$

$$\operatorname{Re}[(\pi |n| / c)\lambda_{|n|}] < \operatorname{Re}\{[\pi(|n| + 1) / c]\lambda_{|n|+1}\}$$

we find that

$$Pe^{-ik\beta y} = ae^{-ik\alpha x} + be^{ik\alpha x} + O(e^{-\sigma|x|}) \quad \text{for} \quad |x| \to \infty$$

$$\sigma = \frac{\pi}{c} \min \operatorname{Re}\left[\left(1 \pm \beta\frac{kc}{\pi}\right)^2 - \left(\frac{kc}{\pi}\right)^2\right]^{-1/2} \qquad (|k| \leqslant \varepsilon < \pi/2c)$$

On the basis of the above, we shall seek the functions P_1 and P_2 in the following form respectively:

$$P_1 = -\rho_1 e^{ik_1\beta_1 y}[e^{-ik_1\alpha_1 x} + V(k_1, \alpha_1, \beta_1)e^{ik_1\alpha_1 x} + u(x, y; k_1)] \tag{7}$$
$$P_2 = -\rho_2 e^{ik_2\beta_2 y}[W(k_2, \alpha_2, \beta_2)e^{-ik_2\alpha_2 x} + v(x, y; k_2)]$$

Here u, v, V and W are interrelated by the following equations:

$$\left(\Delta + 2ik_1\beta_1 \frac{\partial}{\partial y} + k_1^2\alpha_1^2\right) u = 0 \quad (x, y) \in D^+$$
$$\left(\Delta + 2ik_2\beta_2 \frac{\partial}{\partial y} + k_2^2\alpha_2^2\right) v = 0 \quad (x, y) \in D^- \tag{8}$$

$$\frac{\partial u}{\partial n} + ik_1\beta_1 \frac{\partial y}{\partial n} u \bigg|_{\gamma^+} = ik_1\alpha_1 \left(e^{-ik_1\alpha_1 x} - Ve^{ik_1\alpha_1 x}\right) \frac{\partial x}{\partial n} - ik_1\beta_1 \left(e^{-ik_1\alpha_1 x} + Ve^{ik_1\alpha_1 x}\right) \frac{\partial y}{\partial n} \bigg|_{\gamma^+}$$

(γ^+ is that part of γ which corresponds to $x > 0$) (9)

$$\frac{\partial v}{\partial n} + ik_2\beta_2 \frac{\partial y}{\partial n} v \bigg|_{\gamma^-} = ik_2\alpha_2 W e^{-ik_2\alpha_2 x} \frac{\partial x}{\partial n} - ik_2\beta_2 W e^{-ik_2\alpha_2 x} \frac{\partial y}{\partial n} \bigg|_{\gamma^-} \tag{10}$$

(γ^- is that part of γ which corresponds to $x < 0$)

$$u = O(e^{-\sigma x}) \text{ for } x \to \infty, \; v = O(e^{\sigma x}) \text{ for } x \to -\infty \tag{11}$$

$$u(x, y - c) = u(x, y + c), \quad v(x, y - c) = v(x, y + c) \tag{12}$$

$$\rho_1(1 + V + u) = \rho_2(W + v) \exp\left[i(k_2\beta_2 - k_1\beta_1)y\right]_L \tag{13}$$

$$ik_1\alpha_1(1 - V) - \frac{\partial u}{\partial x} = \left(ik_2\alpha_2 W - \frac{\partial v}{\partial x}\right) \exp\left[i(k_2\beta_2 - k_1\beta_1) y\right] \bigg|_L \tag{14}$$

$$2ic\left[k_1\alpha_1(1 - V) - k_2\alpha_2 W\right] = -\frac{1}{\rho_1} \int\limits_{\gamma_0^+} P_1 \frac{\partial \Phi_1}{\partial n} dl - \frac{1}{\rho_2} \int\limits_{\gamma_0^-} P_2 \frac{\partial \Phi_2}{\partial n} dl \tag{15}$$

$$2ck_1k_2\alpha_1\alpha_2\left[\rho_1(1 + V) - \rho_2 W\right] = -k_2\alpha_2 \int\limits_{\gamma_0^+} P_1 \frac{\partial \Psi_1}{\partial n} dl - k_1\alpha_1 \int\limits_{\gamma_0^-} P_2 \frac{\partial \Psi_2}{\partial n} dl \tag{16}$$

Here γ_0^+ and γ_0^- are parts of the boundary of one grid element in the strip $|y| < c$, and functions Φ_1, Φ_2, Ψ_1 and Ψ_2 are determined by the following equations:

$$\Phi_1 = \cos(k_1\alpha_1 x)\, e^{-ik_1\beta_1 y}, \quad \Phi_2 = \cos(k_2\alpha_2 x)\, e^{-ik_2\beta_2 y}$$
$$\Psi_1 = \sin(k_1\alpha_1 x)\, e^{-ik_1\beta_1 y}, \quad \Psi_2 = \sin(k_2\alpha_2 x)\, e^{-ik_2\beta_2 y} \tag{17}$$

Equations (8), boundary conditions (9) and (10), periodicity condition (12), and conditions (13) and (14) on L, can be all obtained by substituting P_1 and P_2 from (7) into relations (1)–(5).

To derive Eq. (15), all that is needed is to apply twice the Green formula, respectively, for functions P_1 and over the domain D_0^+(D_0^+ is that part of D^+ which lies in the strip $|y| < c$) and for functions P_2 and Φ_2 over the domain D_0^- (D_0^- is that part of D^- which lies in the strip $|y| < c$)

$$-2i\rho_1 k_1\alpha_1 c(1 - V) = \int\limits_{\gamma_0^+} P_1 \frac{\partial \Phi_1}{\partial n} dl - \int\limits_{L_0} \frac{\partial P_1}{\partial x} e^{-ik_1\beta_1 y} dy \tag{18}$$

$$2i\rho_2 k_2\alpha_2 c W = \int\limits_{\gamma_0^-} P_2 \frac{\partial \Phi_2}{\partial n} dl + \int\limits_{L_0} \frac{\partial P_2}{\partial x} e^{-ik_2\beta_2 y} dy \tag{19}$$

(L_0 is that part of L which corresponds to $|y| < c$).

Dividing now both sides of (18) by ρ_1 and both sides of (19) by ρ_2, adding the results and allowing for (5) and for

$$k_1\beta_1 = k_2\beta_2 \tag{20}$$

we finally obtain Eq. (15).

Applying the Green formula for the functions P_1 and Ψ_1 over the domain D^+_0 and for the functions P_2 and Ψ_2 over the domain D^-_0, we derive (16) in the same manner as (15).

Since u and v are periodic functions, all further analysis can be confined to one interval only, for example $D_0 = D^+_0 \, V D^-_0$.

Expanding u, v, V and W into series with respect to k in the vicinity of $k = 0$

$$u = \sum_{p=1}^{\infty} u_p k_1^p, \qquad v = \sum_{p=1}^{\infty} v_p k_2^p, \qquad V = \sum_{p=0}^{\infty} V_p k_1^p, \qquad W = \sum_{p=0}^{\infty} W_p k_2^p \tag{21}$$

and substituting (21) into (8)–(16), having first expanded all functions on the right-hand sides into power series with respect to k in the vicinity of $k = 0$, then gathering all terms containing the same power of k and equating their sum to 0, we obtain a recurrent sequence of boundary value problems for Laplace and Poisson equations. In this paper we shall only consider the first boundary value problem of this recurrent sequence, namely:

$$\Delta u_1 = 0, \text{ when } (x, y) \in D^+_0, \quad \Delta v_1 = 0, \text{ when } (x, y) \in D^-_0 \tag{22}$$

$$\left.\frac{\partial u_1}{\partial n}\right|_{\gamma_0^+} = i\alpha_1 (1 - V_0)\frac{\partial x}{\partial n} - i\beta_1 (1 + V_0)\left.\frac{\partial y}{\partial n}\right|_{\gamma_0^+} \tag{23}$$

$$\left.\frac{\partial v_1}{\partial n}\right|_{\gamma_0^-} = i\alpha_2 W_0 \frac{\partial x}{\partial n} - i\beta_2 W_0 \left.\frac{\partial y}{\partial n}\right|_{\gamma_0^-} \tag{24}$$

$$u_1 = O(e^{-(\pi/c)\, x}) \quad \text{for} \quad x \to \infty, \; v_1 = O(e^{(\pi/c)\, x}) \text{ for } \quad x \to -\infty \tag{25}$$

$$u_1(x, c) = u_1(x, -c), \quad v_1(x, c) = v_1(x, -c) \tag{26}$$

$$\rho_1 (V_1 + u_1)\,|_{L_0} = t\rho_2 (W_1 + v_1)\,|_{L_0}, \qquad \left.\frac{\partial u_1}{\partial x}\right|_{L_0} = t\left.\frac{\partial v_1}{\partial x}\right|_{L_0} \tag{27}$$

where

$$t = \beta_1 / \beta_2 = c_1 / c_2$$

Let us note that Eqs. (27) are derived from (13) and (14) with relation (20) and zero approximations of expressions (15) and (16) taken into account. These zero approximations make it also possible to write out the explicit formulas for V_0 and W_0 as follows:

$$V_0 = \frac{\alpha_1\rho_2 - t\alpha_2\rho_1}{\alpha_1\rho_2 + t\alpha_2\rho_1}, \qquad W_0 = \frac{2\alpha_1\rho_1}{\alpha_1\rho_2 + t\alpha_2\rho_1}, \qquad \alpha_2 = \left(1 - \frac{\beta_1^2}{t^2}\right)^{1/2}$$

It is now clear that V_0 and W_0 are the reflection and transmission coefficients, respectively, for the wave P_0 incident on the free (i.e. without grid) interface of the two media.

First approximations of relations (15) and (16) yield the following set of equations for V_1 and W_1: $2ic \;\; (\alpha_1 V_1 + t^2\alpha_2 W_1) = {}^1\!/_2\,[\alpha_1^2(1 + V_0) + t^2\alpha_2^2 W_0]S + i(\beta_1\mu_1 + t^2\beta_2\mu_2)$

$$2d(\rho_1 V_1 - t\rho_2 W_1) = -{}^1\!/_2 i[\rho_1\alpha_1(1 - V_0) + t\rho_2\alpha_2 W_0]S + \rho_1\lambda_1 + t\rho_2\lambda_2 \tag{28}$$

where S is the surface area of one grid element, $2d$ is the length of L_0, and

$$\mu_1 = \int_{\gamma_0^+} u_1 \frac{\partial y}{\partial n}\, dl, \quad \mu_2 = \int_{\gamma_0^-} v_1 \frac{\partial y}{\partial n}\, dl, \quad \lambda_1 = \int_{\gamma_0^+} u_1 \frac{\partial x}{\partial n}\, dl, \quad \lambda_2 = \int_{\gamma_0^-} v_1 \frac{\partial x}{\partial n}\, dl$$

With a view to compute the above functionals, we shall investigate two functions, φ and ψ, periodic with a period of $2c$ and harmonic in D_0, which satisfy the following conditions:

$$\left.\frac{\partial \varphi}{\partial n}\right|_{\gamma_0} = \left.\frac{\partial x}{\partial n}\right|_{\gamma_0}, \qquad \frac{\partial \psi}{\partial n} = \left.\frac{\partial y}{\partial n}\right|_{\gamma_0}, \qquad \gamma_0 = \gamma_0^+ \cup \gamma_0^-$$

$$\varphi = O(e^{-\pi|x|/c}), \; \psi = O(e^{-\pi|x|/c}) \quad \text{for} \quad |x| \to \infty$$

From the Green formula for the functions φ and u_1 over domain D_0^+, and for φ and

v_1 over domain D_0^-, we find that

$$\lambda_1 = \int\limits_{\gamma_0^+} \varphi \frac{\partial u_1}{\partial n} dl - \int\limits_{L_0} u_1 \frac{\partial \varphi}{\partial x} dy + \int\limits_{L_0} \varphi \frac{\partial u_1}{\partial x} dy$$

$$\lambda_2 = \int\limits_{\gamma_0^-} \varphi \frac{\partial v_1}{\partial n} dl + \int\limits_{L_0} v_1 \frac{\partial \varphi}{\partial x} dy - \int\limits_{L_0} \varphi \frac{\partial v_1}{\partial x} dy$$

The function φ is odd with respect to x and, therefore, is equal to 0 on L_0. Adding the last two equations, with (23), (24) and (27) taken into account, we obtain the following relation:

$$\rho_1\lambda_1 + t\rho_2\lambda_2 = -{}^1/_2 i[\rho_1\alpha_1(1 - V_0) + t\rho_2\alpha_2 W_0]\lambda_x - 2(c-d)[\rho_1 V_1 - t\rho_2 W_1] \quad (29)$$

where λ_x is the coefficient of the apparent mass which, according to Sedov [2], is determined by the equality

$$\lambda_x = -\int\limits_{\gamma_0} \varphi \frac{\partial x}{\partial n} dl$$

In the derivation of (29) we have taken into account the symmetry of the grid with respect to x-and y-axes.

Similarly as before, we obtain the expression for the functionals μ_1 an μ_2,

$$i(\beta_1\mu_1 + t^2\beta_2\mu_2) = -{}^1/_2\beta_1^2(1 + V_0 + W_0)\lambda_y, \quad \lambda_y = -\int\limits_{\gamma_0} \psi \frac{\partial y}{\partial n} dl \quad (30)$$

Here λ_y is the coefficient of the apparent mass [2].

Let us now substitute (29) and (30) into (28)

$$\alpha_1 V_1 + t^2\alpha_2 W_1 = {}^1/_4 ic^{-1}\{(1 + V_0 + W_0)\beta_1^2\lambda_y - [\alpha_1^2(1 + V_0) + t^2\alpha_2^2 W_0]S\}$$

$$\rho_1 V_1 - t\rho_2 W_1 = -{}^1/_4 ic^{-1} [\rho_1\alpha_1(1 - V_0) + t\rho_2\alpha_2 W_0](S + \lambda_x)$$

From this, for $d > 0$ and $\alpha_1 \neq 0$ we finally obtain

$$V_1 = \frac{i}{4c(\alpha_1\rho_2 + t\alpha_2\rho_1)}\{\rho_2[(1 + V_0 + W_0)\beta_1^2\lambda_y - (\alpha_1^2(1 + V_0) + t^2\alpha_2^2 W_0)S] - t\alpha_2[\rho_1\alpha_1(1 - V_0) + t\rho_2\alpha_2 W_0](S + \lambda_x)\}$$

$$W_1 = \frac{i}{4ct(\alpha_1\rho_2 + t\alpha_2\rho_1)}\{\rho_1[(1 + V_0 + W_0)\beta_1^2\lambda_y - (\alpha_1^2(1 + V_0) + t^2\alpha_2^2 W_0)S] + \alpha_1[\rho_1\alpha_1(1 - V_0) + t\rho_2\alpha_2 W_0](S + \lambda_x)\}$$

We have thus determined the asymptotic behavior of the sonic pressure field for a plane wave P_0 incident on a fine rigid grid; its form is as follows:

$$P_1 \sim -\rho_1 P_0 - \rho_1[V_0 + k_1 V_1 + O(k_1^2)]e^{ik_1(\alpha_1 x + \beta_1 y)} \qquad \text{for } x \to \infty,\ k_1 \to 0$$

$$P_2 \sim -\rho_2[W_0 + k_2 W_1 + O(k_2^2)]e^{-ik_2(\alpha_2 x - \beta_2 y)} \qquad \text{for } x \to \infty,\ k_2 \to 0$$

Thus, when the apparent mass of the grid is known, we can compute the reflection and transmission coefficients of the grid.

Finally, when the particular case of normal incidence of a plane wave on the grid is considered under the conditions $\rho_1 = \rho_2$ and $c_1 = c_2$, we obtain results which are in agreement with those produced by Gurevich [3].

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Translated by J. M. S.

ON THE EXISTENCE AND VELOCITY OF PROPAGATION OF NONLINEAR STEADY WAVES

PMM Vol. 34, №3, 1970, pp. 548-552
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(Received November 4, 1968)

The problems of existence and of the upper bound of the velocity of propagation of simple steady waves for the nonlinear wave equation which arises particularly in the analysis of signal transmission in an active RCL line are investigated. It is shown that simple steady waves do exist under certain conditions which the parameters of the nonlinear medium (the line parameters) must satisfy and that the velocity of propagation of these waves does not exceed a certain value which is strictly smaller than the limiting wave propagation velocity in the medium.

The investigation of simple steady waves in nonlinear media associated either with the asymptotic transition of the system from one equilibrium state to another or with return to the initial state is of great practical importance. We need merely point to such physical phenomena as the propagation of a normal combustion front [1], excitation in a neuristor line [2], and a whole series of processes in distributed semiconductor systems such as the Gunn effect [3].

Let us consider the nonlinear wave equation

$$D\left[\frac{1}{s^2}\frac{\partial^2 c}{\partial t^2}-\frac{\partial^2 c}{\partial x^2}\right]+\left[1-\frac{D}{s^2}\frac{dQ}{dc}\right]\frac{\partial c}{\partial t}=Q(c)\qquad (D=\text{const}) \tag{1}$$

where s is the limiting wave propagation velocity and $Q(c)$ is the nonlinear "source".

As $s\to\infty$ Eq. (1) degenerates into the nonlinear diffusion equation

$$\frac{\partial c}{\partial t}=D\frac{\partial^2 c}{\partial x^2}+Q(c) \tag{2}$$

As noted above, wave equation (1) can be arrived at by analyzing signal transmission in an active RCL transmission line described by a system of nonlinear telegraphic equations for the form

$$-\frac{\partial\varphi}{\partial x}=Rj+L\frac{\partial j}{\partial t},\qquad -\frac{\partial j}{\partial x}=C\frac{\partial\varphi}{\partial t}+J(\varphi) \tag{3}$$

where R, C and L are, respectively, the resistance, capacitance, and inductance per unit length and $J(\varphi)$ is the nonlinear leakage current. System (3) defines the distribution